STRING THEORY AND MATRIX MODELS [a]

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It is generally accepted that the double-scaled 1D matrix model is equivalent to the $c = 1$ string theory with tachyon condensation. There remain however puzzles that are to be clarified in order to utilize this connection for our quest towards possible non-perturbative formulation of string theory. We discuss some of the issues that are related to the space-time interpretation of matrix models, in particular, the questions of leg poles, causality, and black hole background. Finally, a speculation about a possible connection of a deformed matrix model with the idea of Dirichret brane is presented.

1 String theory from matrix models

There are many reasons for believing that in string theory non-perturbative effects play crucial roles at various stages in constructing reasonable solutions. First of all, the space (moduli space) of the perturbative vacua of string theory is so rich that its structure should be formulated as a sort of generalized dynamical theory beyond perturbation theory. More importantly, general string perturbation series with respect to string coupling constant $g_{\rm st}$ are badly divergent (non Borel summable), in fact worse than those in renormalized perturbation series in local field theories. This is so in spite of an important fact that each term of the series is ultraviolet finite in contrast with local field theory, especially, that of quantized Einstein theory. A phenomenon closely related with this is [1] that a typical non-perturbative effect is expected to be of the form $\exp -1/g_{\rm st}$ instead of $\exp -1/g_{\rm st}^2$. Both of these can be seen clearly in string theory interpretation of matrix models. Furthermore, we expect that, if non-perturbative effects are so crucial in string theory, the theory is not even of the theory of strings but is described by more fundamental degrees of freedom. In this respect, matrix models are quite suggestive, since the original degrees of freedom here are infinite dimensional matrix fields which are defined **locally** in target space.

The basic reason why we can expect that matrix models may be interpreted as string theory [2] comes from the random surface interpretation. The continuum limit of random surfaces is described by 2D Liuoville gravity in which the target space of the matter fields is just the base space of the matrix fields. The Liouville degree of freedom can in turn be regarded as an additional target

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space coordinate, provided appropriate background condensations of dilaton Φ and/or tachyon T string modes are assumed.

The so-called linear dilaton background is the simplest exact vacuum solution for string theory with 2D flat space-time background.

$$\Phi = \frac{2}{\sqrt{\alpha'}} x^1, \quad T = 0, \quad G_{\mu\nu} = \eta_{\mu\nu} \tag{1}$$

where x^1 is the spatial coordinate. [b] The string coupling constant is determined by the dilaton condensation as $g_{\rm st} \propto \exp\Phi \sim \exp 2x^1/\sqrt{\alpha'}$. This implies that the perturbation theory around this vacuum cannot be well defined, since the coupling grows indefinitely as $x^1 \to \infty$. Fortunately, however, the linear dilaton vacuum can be deformed without violating conformal invariance by letting tachyon condense in the form

$$T = \mu \mathrm{e}^{2x^1/\sqrt{\alpha'}}. \tag{2}$$

The infinite repulsive wall formed by the tachyon condensation in the region $x^1 \to \infty$ prevents strings going into the strong-coupling region and saves the above difficulty. In the opposite asymptotic region where $x^1 \to -\infty$, the string coupling is exponentially vanishing. Hence strings are appreciably interacting only in the wall region and overall coupling strength is proportional to $\frac{1}{\mu}$, as can be easily seen by a scaling argument shifting the origin of the Liouville coordinate x^1. The asymptotic particle content of 2D string theory is therefore exactly described by the free string theory in the linear dilaton vacuum. Only particle mode which can be adopted as asymptotic states for scattering experiment is the so-called massless tachyon. Other possible physical modes called discrete states can be interpreted as global degrees of freedom whose condensation may further deform the structure of the vacuum solution itself. The existence of discrete states is intimately related with another characteristic feature of the 2D string theory. Namely, the discrete states can be associated with an infinite number of conserved currents which form the W_∞ algebra.

Now let us turn to 1D matrix model. The action for an Hermitian $N \times N$ matrix $M(t)$ is

$$S = \int dt \frac{N}{2} \operatorname{Tr}(\dot{M}(t)^2 + M(t)^2). \tag{3}$$

The inverted harmonic oscillator potential is treated with some finite-volume cutoff at $\lambda = -A$ and taking the limit $A \to \infty$ limit together with $N \to \infty$ afterwards, where λ is the eigenvalue of the matrix M. If we assume that the

[b] Our target space metric here is Minkowski.

physical states of the model are $U(N)$ singlet, the model is equivalent with a system of N non-interacting fermions in the potential $V(\lambda) = -\frac{1}{2}\lambda^2$. Then the ground state is parametrized by the fermi energy e_F measured downward from the top of the potential. In the limit $N \to \infty$, excited states are described as a collective mode of the surface excitation of the fermi sea. In the asymptotic region $|\lambda| \to \infty$, it is a free massless field. The effective field theory (collective field theory) for this mode shows that the overall strength of interaction is given by $\frac{1}{Ne_F}$. Furthermore, it turns out that the tree and one-loop contributions to the ground state energy has logarithmic dependence $\log e_F$ apart from the usual dependence $(Ne_F)^{2-2p}$ for genus p contribution. Under the identification $\mu \sim Ne_F$, these properties are just the behavior expected from the presence of linear-dilaton and tachyon background and strongly suggest the equivalence of the model with 2D string theory [3]. In particular, the massless collective excitation is then identified with the massless tachyon of 2D string. The infinite wall formed by the tachyon condensation for the latter case is replaced by the potential wall for the former with positive μ.

However the final and only legitimate handle for making more precise identification between the matrix model and string theory is the S-matrix. In the matrix model, the S-matrix element can be rigorously calculated to all orders with respect to the strength of interactions [4]. On the other hand, the corresponding computation in string theory is very difficult, because of the presence of tachyon condensation. It requires precise determination of correlation functions for the quantum Liouville theory, which has been an open question for more than a decade. One possible approach to this problem has been the method of *analytic continuation* with respect the number of the insertions of the tachyon condensation operator $\mu e^{2x^1/\sqrt{\alpha'}}$. The result of such computation [5] indicates that the string-theory S-matrix element $A_{\rm string}$ are proportional to the matrix-model S-matrix element $\overline{A}_{\rm matrix}$ apart from the energy(ω)-dependent phase factor $\ell(\omega)$ for each external leg. Although they are pure phases, the leg factors, being energy dependent, cannot be discarded since they affect the space-time trajectories of the strings.

$$
\begin{aligned}
& A(\omega_1, \ldots, \omega_n \to \omega_{n+1}, \ldots, \omega_{n+m})_{\rm string} \\
= \quad & \{\prod_{i=1}^{n+m} \ell(\omega_i)\}\overline{A}(\omega_1, \ldots, \omega_n \to \omega_{n+1}, \ldots, \omega_{n+m})_{\rm matrix}
\end{aligned} \tag{4}
$$

where we are considering the n(in) $\to$ m(out) scattering and

$$
\ell(\omega) = \mu^{-\sqrt{\alpha'}i\omega/2}\frac{\Gamma(i\sqrt{\alpha'}\omega)}{\Gamma(-i\sqrt{\alpha'}\omega)}. \tag{5}
$$

The matrix model S-matrix $\overline{A}_{\text{matrix}}$ have a genus expansion,

$$\overline{A}_{\text{matrix}} = \frac{1}{\mu^{m+n-2}}(a_0(\sqrt{\alpha'}\omega) + \frac{1}{\mu^2}a_1(\sqrt{\alpha'}\omega) + \cdots) \tag{6}$$

where the genus$-p$ amplitude a_p is a piecewise polynomial with respect to the energies and has no cut and/or pole singularities usually expected for the perturbative S-matrix elements. Higher genus contributions get additional powers of energies as $(\omega/\mu)^{2p}$. Another important property is that a_p always vanishes for vanishing energies for $n+m>2$, even if the kinematical energy factors coming from the relativistic normalization for the wave function are separated out.

Thus the singularities of the string S-matrix are only contained in the leg factor $\ell(\omega)$. This peculiar singularity structure of 2D string theory is a consequence of the very special kinematics of asymptotic massless tachyons in the linear dilaton background. Although we have no complete proof for the correctness of the above S-matrix, there are further evidences which strengthen the result. For instance, the above structure is consistent with the Ward-like identities associated with the W_∞ currents. We refer the audience to Hamada's talk [6] in this meeting about a derivation of the S-matrix using the W_∞ Ward identities.

2 Bulk versus wall scattering: Problem of causality?

A simple but important consequence of the above structure of the S-matrix is that the local-field limit, namely, the zero-slope limit ($\alpha' \to 0$) is trivial for 2D strings. The S-matrix element vanishes except for the 2-point amplitude which is one apart from a normalization factor. In contrast with the cases of critical strings, there is no nontrivial local-field limit in the systematic expansion with respect to α'. This is natural if we remember that the effective string coupling $\exp 2x^1/\sqrt{\alpha'}$ just becomes an infinite vertical wall at the origin $x^1=0$ and for $x^1<0$ the interaction vanishes. In this sense, all nontrivial properties of 2D strings should be understood as a consequence of string extension in spite of the fact that there is no transverse extension of strings.

Keeping this in mind, let us first try to interpret the singularities of the leg factor from the point of view of ordinary local field theory. We first notice that the positions $\omega = \frac{in}{\sqrt{\alpha'}}$ $(n=1,2,\ldots)$ of the poles in the leg factor coincide with those of the discrete states. In fact, the operator product expansion of the vertex operators leads to these poles as in the ordinary critical string theories, if we neglect the effect of tachyon condensation. These poles can also be interpreted as being due to the resonance between incident waves with the

tachyon background which has a pure imaginary momentum $p = i2/\sqrt{\alpha'}$. For example, for n $\to$ 1 scattering with pure imaginary energies, the resonance condition with t insertions of tachyon condensation is

$$\sqrt{\alpha'}\omega_{n+1} = i(n+t-1) = \sum_{i=1}^{n} \omega_i. \tag{7}$$

Furthermore, in the tree approximation, the residue at these poles are precisely given by the string amplitudes for the linear dilaton background with t-insertions of tachyon condensation operators. In particular, the amplitudes for $t = 0$, sometimes called bulk amplitudes, are just the tree S-matrix elements for the linear-dilaton vacuum without tachyon background. Natsuume and Polchinski [7] interpreted this phenomenon from a view point of classical space-time physics as follows. Fourier-transforming with respect to ω_{n+1} (outgoing) and taking the early time limit $x^0_{n+1} \to -\infty$, we find its leading behavior is given as

$$\begin{aligned} &A_f(x^0_{n+1} + x^1_{n+1}) \\ &\equiv \int_{-\infty}^{\infty} d\omega_{n+1} \overline{f}(\omega_{n+1}) A(\omega_{n+1}) \mathrm{e}^{i\omega_{n+1}(x^0_{n+1}+x^1_{n+1})} \\ &\mathrm{e}^{(n-1)(x^0_{n+1}+x^1_{n+1})} \times [\text{residue of } A(\omega_{n+1}) \text{ at } \sqrt{\alpha'}\omega_{n+1} = i(n-1)]. \end{aligned} \tag{8}$$

Namely, the leading early-time limit of the string tree amplitudes are proportional to the bulk amplitudes. This is natural since in the early-time limit, the string interaction is occurring only in the asymptotic region of space-time, *provided* that the incident wave packets are sufficiently localized. Thus the bulk amplitudes are exponentially small tails of the string amplitudes, whose main contributions are actually wall contributions.

From the matrix-model point of view, the leg factor is a nonlocal field redefinition for the massless tachyon. Its early time $x^0 \pm x^1 \to -\infty$ behavior is

$$\tilde{\ell}(x^0 \pm x^1) \equiv \int_{-\infty}^{\infty} \omega \mathrm{e}^{i\omega(x^0 \pm x^1)} \ell(\omega) \sim \mathrm{e}^{(x^0 \pm x^1)/\sqrt{\alpha'}}. \tag{9}$$

In the free-fermion picture, on the other hand, what is occurring is simply a potential scattering. A comparison of (9) with (8) shows that the exponential tail is produced entirely by the field redefinition corresponding to the leg factor. Furthermore, to reproduce precisely the bulk amplitudes, it is crucial [8] that the W_∞ charges are conserved in the potential scattering. This then raises a subtle question about causality in the matrix model interpretation of string scattering when the incident wave is not small such that the height of the

incident wave exceeds the top of the wall. Since the W_∞ charges are not conserved in this case, we cannot have correct bulk scattering from the matrix model. On the other hand from the usual space-time view point, even if the amplitude is large, the interaction can be arbitrarily small in the asymptotic region, and the perturbative approximation for the buld amplitude should be valid provided the wave packet is sufficiently localized, since then the wave packet do not reach to the wall region. Does this indicate [8] that the matrix-model interpretation of string theory necessarily violate causality? I would like to add three remarks related with this question, although they do not answer the question directly.

1. The classical space-time picture explained above is only valid if we neglect the positivity of energy in quantum theory. I will argue that the bulk amplitudes cannot be the leading behavior even in the early time limit for the system with massless particle.

2. We should take into account the effect of string extension, since, as I have emphasized before, all nontrivial behaviors of the string amplitudes should actually be consequences of string extension.

3. Finally, there is a natural modification of the matrix model potentials such that the W_∞ charges are conserved even for large amplitudes. It is plausible that the modified model describes a black hole background.

In the remainder of this section, the first point will be discussed. I will comment on the second and third points later.

It is well known that if only the S-matrix elements are given, we can only talk about, at best, the so-called *macro causality.* Namely, causality is valid only within exponentially small errors, because of the impossibility of localizing particle positions in relativistic quantum theory. For theories with massless particles, the situation is worse since we can only localize the wave packet with power behaving tails. Consider a wave packet of out-going asymptotic state

$$\psi(x^0+x^1) = \int_0^\infty d\omega f(\omega) \mathrm{e}^{-i\omega(x^0+x^1)} \tag{10}$$

where $f(\omega)$ is peaked around some value ω_0. Here it is crucial to note that the range of integration for ω is the positive real axis, since negative energy is not allowed for the asymptotic particle states. Assuming that $f(\omega)$ is analytic near $\omega = 0$ but vanishes rapidly for $\omega \to i\infty$, and has a pole at $\sqrt{\alpha'}\omega = i$,

$$\psi(x^0+x^1) \sim -\int_{-\infty}^0 d\omega f(\omega) \mathrm{e}^{-i\omega(x^0+x^1)} + 2\pi i \mathrm{Res} f(\omega)\Big|_{\sqrt{\alpha'}\omega=i} \times \mathrm{e}^{x^0+x^1} \tag{11}$$

But a general theorem says

$$\int_{-\infty}^{0} d\omega f(\omega) \mathrm{e}^{-i\omega(x^0+x^1)} \sim \mathrm{O}(\frac{1}{|x^0+x^1|^\eta}) \tag{12}$$

as $x^0 \to -\infty$ for some $\eta \geq 1$ where η is determined by the behavior of $f(\omega)$ near the origin $\omega = 0$. For example, $\eta = 1$ for $f(\omega) \sim$ non-zero constant at $\omega = 0$. Thus the leading term of $\tilde{A}(x^0 + x^1)$ in the early-time limit is suppressed only by a power behaving term. This shows that the bulk amplitudes can never be clearly separated even in the early-time limit in massless theories. In the space-time picture, we can easily imagine that such a power-behaved contribution comes from processes associated with the pair creation of antiparticles near the wall. Remember that the apparent causality violation in quantum field theory is in general due to the existence [c] of antiparticles, corresponding to particles traveling backward in time. Causal propagator gives power behaving contribution outside light cone for massless fields.

3 String extension and scattering phase shift

The effect of string extension would make the issue of causality even more subtle. Unfortunately, however, we do not have appropriate space-time formulation of 2D string theory which properly takes into account the string extension. Here, I briefly describe a sample calculation [10] of scattering phase shift, to exhibit a dramatic role of string extension.

Consider 2-point amplitude $A(\omega \to \omega)$ in the usual world sheet picture,

$$S = \frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{\hat{g}} \Big\{ \hat{g}^{ab} \partial_a X_\nu \partial_b X^\nu - 2\sqrt{\alpha'}\, \hat{R}^{(2)} x^1 + \mu\, \mathrm{e}^{2x^1/\sqrt{\alpha'}} \Big\} \tag{13}$$

with vertex operators

$$T_{\pm\omega} = e^{-i\omega x^0(z,\bar{z})}\, \mathrm{e}^{(-\frac{2}{\sqrt{\alpha'}} \pm i\omega) x^1(z,\bar{z})}. \tag{14}$$

Simple perturbative expansion with respect to α' does not work, nor is with respect to μ. Fortunately, however, in the limit of $\omega \to \infty$, the one-loop approximation gives a correct result. Perform standard semi-classical calculation by making a shift $x^1 \to x^1_{\mathrm{classical}} + \tilde{x}^1$. Classical (WKB) amplitude is given by

$$\mathrm{e}^{iS_{\mathrm{classical}}}, \tag{15}$$

$$S_{\mathrm{classical}} = -\sqrt{\alpha'}\omega \ln(\mu/2) + 2\sqrt{\alpha'}\omega(\ln(\sqrt{\alpha'}\omega) - 1). \tag{16}$$

[c] See for example Feynman's lecture [9] on why there must be antiparticles.

This just corresponds to the center of mass motion of the string. One-loop contribution corresponding to the longitudinal fluctuation of the string for large ω is

$$e^{iS_{1-\text{loop}}}, \tag{17}$$

$$S_{1-\text{loop}} = 2\sqrt{\alpha'}\omega(\ln(\sqrt{\alpha'}\omega) - 1). \tag{18}$$

The final result $\mathrm{e}^{i(S_{\text{classical}}+S_{1-\text{loop}})}$ coincides with the high-energy limit of the two-point amplitude

$$\mu^{-i\sqrt{\alpha'}\omega}\left(\frac{\Gamma(+i\sqrt{\alpha'}\omega)}{\Gamma(-i\sqrt{\alpha'}\omega)}\right)^2 \tag{19}$$

which is obtained by *analytic* continuation in the number of insertions of tachyon condensation operators.

This clearly shows the double-Gamma structure of the two-point amplitude coming from the leg factors and an essential role of string extension in producing the leg factors. Note that the center of mass motion is not sufficient to describe 2D string dynamics, even though there is no transverse modes.

4 Deformed matrix model, black-hole background and D-brane

Let us next discuss a possible modification of the standard matrix model. In the ordinary $c = 1$ matrix model, the W_∞ charges are conserved only for small amplitudes. There is, however, almost unique modification [11] of the potential which remedies the situation.

$$V(M) = -\frac{1}{2}\mathrm{Tr}M^2 \to \frac{1}{2}\mathrm{Tr}(-M^2 + \frac{m}{M^2}). \tag{20}$$

Scaling property [12] of the deformation term $m\mathrm{Tr}(1/M^2)$ is just consistent with the deformation corresponding to black-hole mass of the conformal-field theory approach. Namely, the string coupling scales as $g_{\text{st}} \sim \frac{1}{\sqrt{m}}$. In CFT approach, this comes from the difference that the x^1 dependence of the black-hole mass operator is $\exp\frac{4x^1}{\sqrt{\alpha'}}$ in contrast to $\exp\frac{2x^1}{\sqrt{\alpha'}}$ of the tachyon condensation corresponding to $g_{\text{st}} \sim \frac{1}{\mu}$.

The deformed model leads to a curious prediction [12] that the odd-point scattering amplitude vanishes. Note that this is not inconceivable since the bulk amplitudes can never be clearly separated as argued above. We may interpret this phenomenon as a consequence of the compactification of time in the Euclidean black hole corresponding to the temperature $T_H = 3\pi\sqrt{\alpha'}$ [13] which leads the discrete Euclidean energies $\omega_n = \frac{2n}{3\sqrt{\alpha'}}$. It is easy to check that

the resonance condition (7) for the insertion of the black hole mass cannot be satisfied for odd point amplitudes. Equivalently, in the Euclidean black hole case, allowed discrete states must have even Euclidean energies $\omega = \frac{2n}{\sqrt{\alpha'}}$. We therefore expect that the leg factor should only exhibit pole singularities at even energies. Related with this is the fact that the W_∞ algebra is also reduced to half of the standard model. The odd energy currents are excluded.

Next I would like to speculate on a new possibility of interpreting the deformation. Recently, various forms of duality relations in string theory is attracting great interest. In particular, Polchinski [14] pointed out that the soliton-type excitations of string theory may be treated dynamically by introducing open strings with Dirichret boundary condition signifying the coupling of strings to certain extended dynamical object called 'Dirichret branes' (D-brane). If the dimensionality of the string coordinates subject to Dirichret condition is p and the boundary condition for the remaining coordinates are Neumann, the dimension of the D-brane is $d-p-1$ in the d-dimensional target space-time.

Let us now consider the deformation term in the form

$$\mathrm{Tr}\frac{1}{M^2} = \int_0^\infty d\ell \, \mathrm{Tr} e^{-\ell M^2}. \tag{21}$$

In the random-surface interpretation, the operator $\mathrm{Tr} e^{-\ell M^2}$ creates a macroscopic hole, roughly, of length $\ell \sim \exp \frac{2x^1}{\sqrt{\alpha'}}$ on the world sheet. This amounts to setting a Dirichret condition for the boundary of the hole with respect to the spatial coordinate x^1. The ℓ-integration means that the position of the corresponding zero-dimensional D-brane must be integrated over with a special weight. If this interpretation is correct, the black hole horizon is somehow replaced by specially weighted D-branes. [d] It is a challenge to establish this intriguing possibility in a more concrete way. To make progress along this line, we need more precise understanding on both the space-time picture of the matrix model and the general dynamics of D-branes, in particular, the origin of the leg factor from the view point of the matrix model and the dynamics of D-branes in the presence of nontrivial dilaton condensation [16]. [e]

To conclude, I would like to emphasize that matrix models have still many facets to be pursued and learned in seeking for possible nonperturbative frame-

[d] After the conference, a very interesting paper [15] appeared pointing out that the black hole entropy may be derived from the D-brane picture for the extremal black hole.

[e] I would like to mention here a possible analogy with the similar phase factor appearing in the scattering amplitudes of QED. It might be worthwhile to pursue this analogy by trying to construct some interacting fermion theory such that the S-matrix for bosonized excitations is just given by the free-fermion S-matrix multiplied with the leg factors. The fermion might then be interpreted as the field describing string solitons.

work of string theory. We must try to achieve unification of matrix-model methods and other approaches. For example, we have to further develop our understanding on the relation of matrix model approach and the string field theories, and on supersymmetric and higher dimensional generalizations of the matrix-model method. It would also be an interesting challenge to formulate string dualities (both T and S dualities) within the matrix-models or perhaps some generalized framework.

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References

1. S. Shenker, in Proc. Random Surfaces and Quantum Gravity, eds. O. Alvarez et al. (Plenum Press, 1991) .
2. For reviews, see, e. g., I. Klebanov, Proc. Trieste Spring School 1991, eds. J. Harvey et al. (World Scientific, Singapore, 1991); J. Polchinski, Les Houshes lecture, hep-th/9411028 (1994).
3. J. Polchinski, Nucl. Phys. B346(1990)253.
4. R. Dijkgraaf, G. Moor and M. Plesser, Nucl. Phys. B425(1991)356.
5. M. Goulian and M. Li, Phys. Rev. Lett. 66 (1991) 2051.
6. K. Hamada, in this volume and references therein.
7. M. Natsuume and J. Polchinski, Nucl. Phys. B424 (1994) 137.
8. J. Polchinski, Phys. Rev. Lett. 74 (1995) 63.
9. R. P. Feynman. in "Elementary Particles and The Laws of Physics" by R. P. Feynman and S. Weinberg (Cambridge University Press, 1987).
10. A. Jevicki, M. Li and T. Yoneya, Nucl. Phys. B448(1995) 277.
11. J. Avan and A. Jevicki, Comm. Math. Phys. 150(1992) 149.
12. A. Jevicki and T. Yoneya, Nucl. Phys. B411(1994) 64.
13. R. Dijkgraaf, E. Verlinde and R. Verlinde, Nucl. Phys. B371(1992) 269.
14. J. Polchinski, hep-th/9510017 (1995) and references therein.
15. C. Vafa and A. Strominger, hep-th/9601029 (1996).
16. M. Li, hep-th/9512042.